Available online at www.sciencedirect.com

SCIENCE DIRECT®

Linear Algebra and its Applications 416 (2006) 773–775

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

A note on definition of matrix convex functions ☆

Oleg E. Tikhonov

Research Institute of Mathematics and Mechanics, Kazan State University, Nuzhina 17, Kazan 420008, Russia

Received 6 May 2005; accepted 20 December 2005
Available online 8 February 2006
Submitted by C. Davis

Abstract

We prove that a real-valued function f defined on an interval S in $\mathbf{R}$ is matrix convex if and only if for any natural k, for all families of positive operators $\{A_i\}_{i=1}^k$ in a finite-dimensional Hilbert space, such that $\sum_{i=1}^k A_i = 1$, and arbitrary numbers $x_i \in S$, the inequality

$$f\left(\sum_{i=1}^{k} x_i A_i\right) \leqslant \sum_{i=1}^{k} f(x_i) A_i$$

holds true.

Keywords: Matrix convex function; The Neumark theorem

The aim of this note is to add one more characterization to the list of known characterizations of matrix and operator convex functions (see, e. g., [1,3,4]).

In what follows, S stands for an interval in $\mathbf{R}$ of an arbitrary type. For a function $f : S \to \mathbf{R}$ and a self-adjoint operator X in a finite-dimensional Hilbert space H with spectrum in S, the value $f(X)$ is defined by the spectral theorem. The identity operator in H is denoted by 1_H.

Theorem. *For a function* $f : S \to \mathbf{R}$, *the following conditions are equivalent*:

(i) f *is matrix convex*, *i.e.*,

$$f(\alpha X + (1-\alpha)Y) \leqslant \alpha f(X) + (1-\alpha) f(Y)$$

☆ Supported by the Russian Foundation for Basic Research (grant no. 05-01-00799).
E-mail address: Oleg.Tikhonov@ksu.ru

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2005.12.019

for all self-adjoint operators X, Y in a finite-dimensional Hilbert space and every $\alpha \in [0, 1]$, provided that spectra of X and Y lie in S;

(ii) *for any natural k, for all families of positive operators $\{A_i\}_{i=1}^k$ in a finite-dimensional Hilbert space H, such that $\sum_{i=1}^k A_i = 1_H$, and arbitrary numbers $x_i \in S$, the inequality*

$$f\left(\sum_{i=1}^k x_i A_i\right) \leqslant \sum_{i=1}^k f(x_i) A_i$$

holds true.

Proof. Since

$$\left(\min_i x_i\right) 1_H \leqslant \sum_{i=1}^k x_i A_i \leqslant \left(\max_i x_i\right) 1_H,$$

the left hand side in the inequality in (ii) is well-defined.

(ii) $\Longrightarrow$ (i). Let H denote the space where the operators X and Y act,

$$X = \sum_{i \in I} \lambda_i P_i \quad \text{and} \quad Y = \sum_{j \in J} \mu_j Q_j$$

be their spectral decompositions. Then

$$\sum_{i \in I} \alpha P_i + \sum_{j \in J} (1 - \alpha) Q_j = 1_H$$

and

$$\begin{aligned} f(\alpha X + (1-\alpha)Y) &= f\left(\sum_{i \in I} \lambda_i \alpha P_i + \sum_{j \in J} \mu_j (1-\alpha) Q_j\right) \\ &\leqslant \sum_{i \in I} f(\lambda_i) \alpha P_i + \sum_{j \in J} f(\mu_j)(1-\alpha) Q_j \\ &= \alpha f(X) + (1-\alpha) f(Y). \end{aligned}$$

(i) $\Longrightarrow$ (ii). It is easy to see that without loss of generality we can suppose that $0 \in S$. By the Neumark theorem [5], there exists a space $\mathscr{H}$ larger than H and a family of mutually orthogonal projections P_i in $\mathscr{H}$ such that $\sum_{i=1}^k P_i = 1_{\mathscr{H}}$ and $A_i = P P_i P|_H$ $(i = 1, \ldots, k)$, where P is the projection from $\mathscr{H}$ onto H. Then we have

$$\begin{aligned} f\left(\sum_{i=1}^k x_i A_i\right) &= f\left(\sum_{i=1}^k x_i P P_i P|_H\right) = f\left(P\left(\sum_{i=1}^k x_i P_i\right) P|_H\right) \\ &= P f\left(P\left(\sum_{i=1}^k x_i P_i\right) P\right) P|_H \\ &\leqslant P f\left(\sum_{i=1}^k x_i P_i\right) P|_H = P\left(\sum_{i=1}^k f(x_i) P_i\right) P|_H \\ &= \sum_{i=1}^k f(x_i) A_i. \qquad \square \end{aligned}$$

(The inequality in the above calculation follows from the Davis inequality [2]: $Pf(PXP)P \leqslant Pf(X)P$.)

Remark. After the first version of the present paper had been submitted, professor T. Ando recommended to add one more equivalent condition:

(iii) for any natural k, for any family of operators $\{B_i\}_{i=1}^k$ with spectra in S and for any family of operators $\{A_i\}_{i=1}^k$ such that $\sum_{i=1}^k A_i^* A_i = 1_H$ the inequality

$$f\left(\sum_{i=1}^{k} A_i^* B_i A_i\right) \leqslant \sum_{i=1}^{k} A_i^* f(B_i) A_i$$

holds.

He gave a proof of the implications (ii) $\Longleftrightarrow$ (iii) (one can read that proof in [6]). Since the equivalence of (i) and (iii) is known (it goes back to [3]), the proof due to Ando gives an alternative proof of the theorem presented in this note. It is worth to note that in contrast to (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii), the proof of (ii) $\Longleftrightarrow$ (iii) does not refer to the Neumark theorem and deals with one and the same finite-dimensional Hilbert space.

Acknowledgments

The author thanks T. Ando for a useful comment on the original version of the paper.

References

[1] J. Bendat, S. Sherman, Monotone and convex operator functions, Trans. Amer. Math. Soc. 79 (1955) 58–71.
[2] C. Davis, A Schwarz inequality for convex operator functions, Proc. Amer. Math. Soc. 8 (1957) 42–44.
[3] C. Davis, Notions generalizing convexity for functions defined on spaces of matrices, Proc. Amer. Math. Soc. Symposia (Convexity) 7 (1963) 187–201.
[4] F. Hansen, G.K. Pedersen, Jensen's inequality for operators and Löwner's theorem, Math. Ann. 258 (1982) 229–241.
[5] M.A. Neumark, On a representation of additive operator set functions, Dokl. Akad. Nauk SSSR 41 (9) (1943) 373–375. (Russian); English translation: C.R. (Doklady) Akad. Sci. USSR. (N.S.) 41 (1943), 359–361.
[6] O.E. Tikhonov, A note on definition of matrix convex functions with a supplement by T. Ando, Preprints of the N.G. Chebotarev Research Institute of Mathematics and Mechanics of Kazan State University, Preprint 0003-2005. Available from: <http://www.niimm.ksu.ru/data/preprints/thepreprints/0003-2005.ps>.